ISRAEL JOURNAL OF MATHEMATICS **163** (2008), 125–138
DOI: 10.1007/s11856-008-0007-4

ENDOMORPHISMS OF MATRIX SEMIGROUPS OVER DIVISION RINGS

BY

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ABSTRACT

Let $\mathbb{D}$ be an arbitrary division ring and $M_n(\mathbb{D})$ the multiplicative semigroup of all $n \times n$ matrices over $\mathbb{D}$. We describe the general form of endomorphisms of $M_n(\mathbb{D})$.

1. Introduction and statement of the main results

Let $\mathbb{D}$ be a division ring and n a positive integer. We denote by $M_n(\mathbb{D})$ the set of all $n \times n$ matrices over $\mathbb{D}$. In this paper we study multiplicative maps $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$, that is, maps satisfying $\phi(AB) = \phi(A)\phi(B)$, $A, B \in M_n(\mathbb{D})$. The problem of characterizing multiplicative maps on matrices over a principal ideal domain was solved by Jodeit and Lam [10]. Pierce [13] showed that their result does not hold for matrices over an arbitrary integral domain. The motivation to study multiplicative maps on matrices over an arbitrary division ring comes from the Wedderburn–Artin theorem [4, p. 44] stating that every simple artinian ring is isomorphic to a matrix ring over a division ring. The methods of Jodeit and Lam do not work in our non-commutative setting. In fact, the starting part of our proof will be the same as in the paper of Jodeit and Lam. After the first step we will use a completely different

* Supported in part by a grant from the Ministry of Science of Slovenia.
Received November 21, 2005

approach based on the observation that the set of idempotents is invariant under every multiplicative map. Thus, our proof depends heavily on some recent results concerning the structure of maps on idempotents preserving order or orthogonality [15].

Let us start with examples of multiplicative maps on $M_n(\mathbb{D})$. Following Jodeit and Lam we will call a multiplicative map $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ degenerate if $\phi(A) = 0$ for every singular $A \in M_n(\mathbb{D})$. Let $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ be a degenerate multiplicative map and denote $\phi(I) = P$. Then P is an idempotent matrix. It follows (see the second section) that there exists an invertible matrix $S \in M_n(\mathbb{D})$ such that

$$SPS^{-1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

where I is the $r \times r$ identity matrix for some integer r, $0 \le r \le n$. If $r = 0$, then clearly, $\phi = 0$. So, assume that $r > 0$. From $\phi(A) = \phi(I)\phi(A)\phi(I)$, $A \in M_n(\mathbb{D})$, we conclude that the map $A \mapsto S\phi(A)S^{-1}$ can be considered as a multiplicative degenerate map from $M_n(\mathbb{D}) \to M_r(\mathbb{D})$ (for each A we take the upper left $r \times r$ corner of $S\phi(A)S^{-1}$; all entries outside this corner are equal to zero) sending the $n \times n$ identity matrix into the $r \times r$ identity matrix. For every invertible $A \in M_n(\mathbb{D})$ we have $\phi(A)\phi(A^{-1}) = I$, and because every singular matrix is mapped into the zero matrix, we have arrived at the problem of characterizing homomorphisms ψ from $GL_n(\mathbb{D})$ into $GL_r(\mathbb{D})$, $r \le n$. Such homomorphisms can be viewed as representations of the full matrix group $GL_n(\mathbb{D})$ in $GL_r(\mathbb{D})$. The theory of group representations is well-developed. The results are highly non-trivial even in the case when $\mathbb{D}$ is a field (see, e.g., [6] and [16]). Let us mention here that the case $r < n$ is much easier than the case $n = r$. If $r < n$, then every homomorphism ψ from $GL_n(\mathbb{D})$ into $GL_r(\mathbb{D})$ is of the form $\psi(A) = \varphi(\det A)$ for some homomorphism $\varphi\colon \mathbb{D}^*/C \to GL_r(\mathbb{D})$ [3]. Here, $\mathbb{D}^*$ is the multiplicative group of all non-zero elements of $\mathbb{D}$, C is the commutator subgroup of $\mathbb{D}^*$, and $\det A$ denotes the Dieudonné's determinant of A (for the definition and some basic properties see [1], [2], and [12]).

Let us continue with non-degenerate examples. If $A = [a_{ij}] \in M_n(\mathbb{D})$ is any matrix and $\sigma\colon \mathbb{D} \to \mathbb{D}$ an endomorphism of division ring $\mathbb{D}$, then we denote by A^σ the matrix obtained from A by applying σ entrywise, $A^\sigma = [a_{ij}]^\sigma = [\sigma(a_{ij})]$. Clearly, for every invertible matrix $T \in M_n(\mathbb{D})$ and every endomorphism σ of

$\mathbb{D}$ the map $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ defined by

$$\phi(A) = TA^\sigma T^{-1}, \quad A \in M_n(\mathbb{D}), \tag{1}$$

is multiplicative.

As above, we denote by C the commutator subgroup of $\mathbb{D}^*$. We adjoin to the group $\mathbb{D}^*/C$ a zero element with the obvious multiplication, and denote the semigroup thus obtained by $\overline{\mathbb{D}}$. Let $\varphi\colon \mathbb{D}^*/C \to GL_p(\mathbb{D})$, $p < n$, be a homomorphism of groups. We extend it to a multiplicative map from $\overline{\mathbb{D}}$ into $M_p(\mathbb{D})$ by defining $\varphi(0) = 0$. Let further $T \in M_n(\mathbb{D})$ be an invertible matrix. Define $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ by

$$\phi(A) = T \begin{bmatrix} \varphi(\det A) & 0 \\ 0 & P \end{bmatrix} T^{-1}, \quad A \in M_n(\mathbb{D}). \tag{2}$$

Here, P denotes an $(n-p)\times(n-p)$ idempotent matrix. Then ϕ is multiplicative. Let us mention here that in the case when $p = 0$ we have the constant map, $\phi(A) = Q$, $A \in M_n(\mathbb{D})$. Here, $Q = TPT^{-1}$.

Finally, let $\mathbb{D} = \mathbb{F}$ be a field. For each $A \in M_n(\mathbb{F})$ we denote by A^* the matrix of its cofactors. Let $T \in M_n(\mathbb{F})$ be an invertible matrix and $\sigma\colon \mathbb{F} \to \mathbb{F}$ an endomorphism of the field $\mathbb{F}$. Then the map $\phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ defined by

$$\phi(A) = T(A^\sigma)^* T^{-1}, \quad A \in M_n(\mathbb{F}), \tag{3}$$

is multiplicative. The question here is whether one can construct in a similar way multplicative maps on matrices over non-commutative division rings. Of course, in the non-commutative setting one would expect that the matrix A^* will be defined in a similar way as in the commutative case with the Dieudonné's determinant instead of the usual determinant. It turns out that the answer is negative. So, the structural result in the non-commutative case is even simpler than in the commutative case.

THEOREM 1.1: *Let $\mathbb{D}$ be a non-commutative division ring and $n \geq 2$ an integer. Assume that $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is a non-degenerate multiplicative map. Then ϕ is either of the form* (1), *or of the form* (2), *where T, P, σ and φ are as above.*

THEOREM 1.2: *Let $\mathbb{D}$ be a division ring and $n \geq 2$ an integer. Assume that $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ is a non-degenerate multiplicative map. Then ϕ is either of the form* (1), *or of the form* (2), *or $\mathbb{D}$ is commutative and ϕ is of the form* (3). *Here, T, P, σ and φ are as above.*

Corollary 1.3: *Let $\mathbb{D}$ be a division ring and n, m integers with $n \geq 2$ and $n > m$. Assume that $\phi\colon M_n(\mathbb{D}) \to M_m(\mathbb{D})$ is a non-degenerate multiplicative map. Then ϕ is of the form* (2), *where $T \in M_m(\mathbb{D})$ is invertible, $\varphi\colon \overline{\mathbb{D}} \to M_p(\mathbb{D})$ is a non-zero homomorphism with $\varphi(0) = 0$, and P is an $(m-p) \times (m-p)$ idempotent matrix, $0 \leq p < m$.*

Let us conclude this section with some brief historical remarks. Automorphisms of matrix semigroups over fields were characterized by Gluskin [5] and Halezov [7, 8]. As already mentioned, these results were generalized by Jodeit and Lam [10] who described the general form of non-degenerate endomorphisms of matrix semigroups over principal ideal domains. The main tool in our proof is a structural result for maps on rank one idempotents preserving orthogonality. The structural problem for such maps is closely related to the problem of describing order-preserving maps on idempotent matrices [15]. Automorphisms of the partially ordered sets of idempotent matrices or operators were characterized by Ovchinnikov [11]. He was mainly interested in the infinite-dimensional case because of applications in physics (see the review MR 95a:46093). In the real matrix case his result was substantionally improved in [14]. This improvement was a main tool in the study of geometry of matrices. Because of further applications in this direction a systematic study of order preserving and orthogonality preserving maps on idempotent matrices over division rings was carried out in [15]. Later on we realized that these results can be applied also to solve the problem treated in this paper.

2. Notation and some preliminary results

Let $\mathbb{D}$ be a division ring, n a positive integer, and $M_n(\mathbb{D})$ the set of all $n \times n$ matrices over $\mathbb{D}$. The symbol E_{ij}, $1 \leq i, j \leq n$, will be used for a matrix having all entries zero except the (i,j)-entry which is equal to 1. By $P_n(\mathbb{D})$ we denote the set of all $n \times n$ idempotent matrices, $P_n(\mathbb{D}) = \{P \in M_n(\mathbb{D}) : P^2 = P\}$.

Let us recall first the definition of the rank of an $n \times n$ matrix A with entries in a division ring $\mathbb{D}$. We will denote by $\mathbb{D}^n$ the set of all $1 \times n$ matrices and consider it always as a left vector space over $\mathbb{D}$. Correspondingly, we have the right vector space of all $n \times 1$ matrices ${}^t\mathbb{D}^n$. We first take the left vector subspace of $\mathbb{D}^n$ generated by the rows of A (the row space of A) and define the row rank of A to be the dimension of this subspace. The column rank of A is the dimension of the right vector space generated by the columns of A. This space is called the column space of A. These two ranks are equal for every matrix over $\mathbb{D}$ and this common value is called the rank of a matrix. If $\operatorname{rank} A = r$ then there exist invertible matrices $T, S \in M_n(\mathbb{D})$ such that

$$TAS = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

Here, I_r is the $r \times r$ identity matrix and the zeroes stand for zero matrices of the appropriate size. In particular, if $\operatorname{rank} A \leq \operatorname{rank} B$, then $A = CBD$ for some $C, D \in M_n(\mathbb{D})$. Rank satisfies the triangle inequality, that is, $\operatorname{rank}(A+B) \leq \operatorname{rank} A + \operatorname{rank} B$ for every pair $A, B \in M_n(\mathbb{D})$ [9, p. 46, Exercise 2]. Note that in general $\operatorname{rank} A$ need not be equal to $\operatorname{rank} {}^tA$. Here, ${}^tA$ denotes the transpose of A. However, if $\tau\colon \mathbb{D} \to \mathbb{D}$ is a non-zero anti-endomorphism of $\mathbb{D}$ (additive map satisfying $\tau(\lambda\mu) = \tau(\mu)\tau(\lambda)$, $\lambda, \mu \in \mathbb{D}$), then $\operatorname{rank} A = \operatorname{rank}^t(A^\tau)$.

Assume that $A, B \in M_n(\mathbb{D})$. Since the multiplication in $\mathbb{D}$ is not necessarily commutative we do not have ${}^t(AB) = {}^tB\ {}^tA$ in general. But if τ is an anti-endomorphism of $\mathbb{D}$ then

$${}^t[(AB)^\tau] = {}^t(B^\tau)\ {}^t(A^\tau).$$

As usual we will identify $n \times n$ matrices with linear operators acting on $\mathbb{D}^n$. Namely, each $n \times n$ matrix A gives rise to a linear operator defined by $x \mapsto xA$, $x \in \mathbb{D}^n$. Then the rank of the matrix A is the dimension of the image $\Im A$ of the corresponding operator A. The kernel of an operator A, $\operatorname{Ker} A = \{x \in \mathbb{D}^n : xA = 0\}$, is the set of all vectors $x \in \mathbb{D}^n$ satisfying $x({}^ty) = 0$ for every ${}^ty$ from the column space of A. Note that $n = \operatorname{rank} A + \dim \operatorname{Ker} A$.

In the sequel we shall need the following fact that is well-known for idempotent matrices over fields and can be also generalized to idempotent matrices over division rings [9, p. 62, Exercise 1]. Assume that $P_1, \ldots, P_k \in P_n(\mathbb{D})$ are pairwise orthogonal, that is, $P_iP_j = P_jP_i = 0$ whenever $i \neq j$. Denote by r_i the rank of P_i. Then there exists an invertible matrix $T \in M_n(\mathbb{D})$ such that for each i, $1 \leq i \leq k$, we have

$$TP_iT^{-1} = \operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$$

where $\operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ is the diagonal matrix in which all the diagonal entries are zero except those in $(r_1 + \cdots + r_{i-1} + 1)$st to $(r_1 + \cdots + r_i)$th rows.

In the proof of our main results we will need the following three simple lemmas.

Lemma 2.1: *Let $A, B, C \in M_n(\mathbb{D})$. Assume that B is of rank one and $ABC = 0$. Then $AB = 0$ or $BC = 0$.*

Proof. As B is of rank one it can be written as $B = ({}^t b)a$ for some non-zero vectors $a \in \mathbb{D}^n$ and ${}^t b \in {}^t\mathbb{D}^n$. It is now clear that $ABC = 0$ implies $A^t b = 0$ or $aC = 0$. ∎

Lemma 2.2: *Let $A, B \in M_n(\mathbb{D})$ be matrices of rank at most one. If* $\mathrm{rank}(I - A - B) = n - 2$*, then A and B are orthogonal rank one idempotents.*

Proof. We have $I = A + B + S$ for some $S \in M_n(\mathbb{D})$ of rank $n-2$. As rank is subadditive both A and B must be of rank one. Identifying matrices with operators we see from $x = xI = xA + xB + xS$, $x \in \mathbb{D}^n$, that $\mathbb{D}^n = \Im A + \Im B + \Im S$. Since the dimensions of these images are one, one, and $n-2$, respectively, we actually have $\mathbb{D}^n = \Im A \oplus \Im B \oplus \Im S$. For $x \in \Im A$ we have $x = xI = xA + xB + xS \in \Im A$, and since the above sum is a direct sum, the vectors xB and xS must be zero, while $xA = x$. Similarly, if $x \in \Im B$ then $xB = x$, and $xA = xS = 0$ and also $x \in \Im S$ yields $x = xS$ and $xA = xB = 0$. Thus, all three operators A, B and S are idempotents and they are pairwise orthogonal. In fact, the same proof shows that if the $n \times n$ identity matrix is the sum of matrices A_j of rank r_j, $j = 1, \ldots, k$, and if $r_1 + \cdots + r_k \le n$, then $r_1 + \cdots + r_k = n$, and the A_j's are pairwise orthogonal idempotents. Results of this type can be found in the literature but as far as we know only for matrices over a field. ∎

Lemma 2.3: *Let $P, Q \in P_n(\mathbb{D})$ be of rank one. Then the following two statements are equivalent:*

- *$PQ = 0$,*
- *$(I - Q)(I - P)$ is of rank $n - 2$.*

Proof. Assume first that $PQ = 0$. Then straightforward computations show that $Q(I - P)$ is an idempotent and that $Q(I - P)$ and P are orthogonal. The matrix $Q(I - P)$ is non-zero, since otherwise we would have $Q = QP$, and consequently, $Q = Q^2 = QPQ = 0$, a contradiction. Thus, the matrix $Q(I - P) + P$, which is a sum of two orthogonal rank one idempotents, is an idempotent of rank two, which further yields that $I - (Q(I - P) + P) = (I - Q)(I - P)$ is of rank $n - 2$.

To prove the other direction assume that $I-Q(I-P)-P$ is of rank $n-2$. Since both matrices $Q(I-P)$ and P are of rank at most one, they are, by the previous lemma, orthogonal idempotents of rank one. In particular, $PQ(I - P) = 0$. By Lemma 2.1, $PQ = 0$. ∎

3. Proof of main results

Let $\mathbb{D}$ be an arbitrary division ring, n an integer satisfying $n \geq 2$, and let $\phi\colon M_n(\mathbb{D}) \to M_n(\mathbb{D})$ be a non-degenerate multiplicative map. Denote by r the first integer such that $\phi(A) \neq 0$ for some $A \in M_n(\mathbb{D})$ with $\operatorname{rank} A = r$. In the first step of our proof (here we follow the idea of Jodeit and Lam) we will prove that either $r = 0$, or $r = 1$, or $r = n - 1$.

If $r = 0$ we are done. So, assume that $0 < r < n$ (in the case $r = n$ our map would be degenerate). Then $\phi(A) = 0$ for every A with $\operatorname{rank} A < r$ and $\phi(A) \neq 0$ for every A with $\operatorname{rank} A \geq r$. Indeed, the first part of this statement is trivial. Let $B \in M_n(\mathbb{D})$ be a matrix of rank r such that $\phi(B) \neq 0$ and A any matrix of rank at least r. Then we already know that there are matrices C and D such that $B = CAD$. So, $\phi(A) = 0$ would imply that $\phi(B) = \phi(C)\phi(A)\phi(D) = 0$, a contradiction. Let $\mathcal{E}$ be the set of all diagonal idempotents E of rank r, that is, the set of all diagonal matrices with r ones and $n - r$ zeroes on the diagonal. The product of any pair of different elements of $\mathcal{E}$ has rank less than r. Thus, we have a system of $\binom{n}{r}$ pairwise orthogonal non-zero idempotents $\{\phi(E) : E \in \mathcal{E}\}$. Pairwise orthogonal idempotents are simultaneously diagonalizable. Hence,

$$\binom{n}{r} \leq n,$$

and consequently, either $r = 1$ or $r = n - 1$, as desired. Moreover, in the case when $r = 1$ we see that E_{11} is mapped into a rank one projection. If B is any rank one matrix, then there exist invertible matrices $T, S \in M_n(\mathbb{D})$ such that $B = TE_{11}S$, which further yields $\phi(B) = \phi(T)\phi(E_{11})\phi(S)$ and $\phi(E_{11}) = \phi(T^{-1})\phi(B)\phi(S^{-1})$. Thus, $\operatorname{rank} \phi(B) = 1$. Similarly, if $r = n - 1$, then every matrix of rank $n - 1$ is mapped into a rank one matrix.

Exactly the same idea yields that if $\xi\colon M_n(\mathbb{D}) \to M_m(\mathbb{D})$, $m < n$, is a multiplicative map with $\xi(0) = 0$, then ξ is degenerate.

So, coming back to multiplicative maps from $M_n(\mathbb{D})$ into itself we have three possibilities. Let us start with the case when $r = 0$. Then $\phi(0)$ is a non-zero idempotent. After composing ϕ with an appropriate similarity transformation $A \mapsto SAS^{-1}$ we may assume that

$$\phi(0) = \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix},$$

where I_p denotes the $p \times p$ identity matrix, $1 \leq p \leq n$. From $\phi(0) = \phi(A)\phi(0) = \phi(0)\phi(A)$, $A \in M_n(\mathbb{D})$, we conclude that there exists a multiplicative map $\xi\colon M_n(\mathbb{D}) \to M_{n-p}(\mathbb{D})$ such that

$$\phi(A) = \begin{bmatrix} \xi(A) & 0 \\ 0 & I_p \end{bmatrix}, \quad A \in M_n(\mathbb{D}).$$

If $p = n$, then $\phi(A) = I$ for every $A \in M_n(\mathbb{D})$. So, let $1 \leq p < n$. By the previous paragraph, ξ is degenerate. The Dieudonné's determinant of any singular matrix A is zero. It is now easy to complete the proof using the description of homomorphisms from $GL_n(\mathbb{D})$ into $GL_k(\mathbb{D}), k < n$, given in the Introduction. Moreover, the above arguments yield also Corollary 1.3. Indeed, under the assumptions of Corollary 1.3 the only possible value of r is $r = 0$.

We continue with the case when $r = 1$. Then every idempotent of rank one is mapped into an idempotent of rank one. For two idempotents $P, Q \in M_n(\mathbb{D})$ of rank one we have $PQ = QP = 0$ if and only if $\phi(P)\phi(Q) = \phi(Q)\phi(P) = \phi(0) = 0$. In other words, the restriction of ϕ to $P_n^1(\mathbb{D})$, the set of all rank one idempotents, considered as a map from $P_n^1(\mathbb{D})$ into itself, preserves orthognality in both directions. It is injective. Indeed, if $\phi(P) = \phi(Q)$ for $P, Q \in P_n^1(\mathbb{D})$, then for every idempotent R of rank one we have $R \perp P$ if and only if $\phi(R) \perp \phi(P) = \phi(Q)$ if and only if $R \perp Q$. It follows easily that $P = Q$. Here we have to distinguish two cases. Assume first that $n \geq 3$. Then by [15, Theorem 4.7], there exist a non-singular matrix $T \in M_n(\mathbb{D})$ and either a non-zero endomorphism $\sigma\colon \mathbb{D} \to \mathbb{D}$ such that

$$\phi(P) = TP^{\sigma}T^{-1}, \quad P \in P_n^1(\mathbb{D}), \tag{5}$$

or a nonzero anti-endomorphism $\tau\colon \mathbb{D} \to \mathbb{D}$ such that

$$\phi(P) = T^t(P^{\tau})T^{-1}, \quad P \in P_n^1(\mathbb{D}).$$

In the second case we have

$$
\begin{aligned}
0 = \phi(0) &= \phi(E_{22}(E_{11} + E_{12})) = \phi(E_{22})\phi(E_{11} + E_{12}) \\
&= TE_{22}T^{-1}T(E_{11} + E_{21})T^{-1} = TE_{21}T^{-1},
\end{aligned}
$$

a contradiction. Thus, this possibility cannot occur. Hence, we have (5), and after replacing ϕ by the map $A \mapsto T^{-1}\phi(A)T$ we may assume that $\phi(P) = P^\sigma$, $P \in P_n^1(\mathbb{D})$. It follows that for every pair $i, j \in \{1, \ldots, n\}$, $i \neq j$, and every $\lambda \in \mathbb{D}$ we have

$$
\begin{aligned}
\phi(\lambda E_{ij}) &= \phi((E_{ii} + \lambda E_{ij})E_{jj}) = \phi(E_{ii} + \lambda E_{ij})\phi(E_{jj}) = (E_{ii} + \sigma(\lambda)E_{ij})E_{jj} \\
&= \sigma(\lambda)E_{ij}.
\end{aligned}
$$

This further yields that

$$\phi(\lambda E_{ii}) = \phi(\lambda E_{ij}E_{ji}) = \sigma(\lambda)E_{ij}E_{ji} = \sigma(\lambda)E_{ii}.$$

Let $A \in M_n(\mathbb{D})$ be an arbitrary matrix and let i, j be any pair of integers, $1 \leq i, j \leq n$. For every $B \in M_n(\mathbb{D})$ we denote by B_{ij} the (i, j)-entry of B. Then

$$
\begin{aligned}
\phi(A)_{ij}E_{ij} &= E_{ii}\phi(A)E_{jj} = \phi(E_{ii})\phi(A)\phi(E_{jj}) = \phi(E_{ii}AE_{jj}) = \phi(A_{ij}E_{ij}) \\
&= \sigma(A_{ij})E_{ij}.
\end{aligned}
$$

Thus, for every $A \in M_n(\mathbb{D})$ we have $\phi(A) = A^\sigma$, as desired.

In the case when $n = 2$ we cannot apply [15, Theorem 4.7]. So, in this case we will give a straightforward proof. The same idea can be applied also in higher dimensions but we decided to include the above argument as it is slightly shorter. We know that $\phi(E_{11})$ and $\phi(E_{22})$ are orthogonal rank one idempotents. After composing ϕ with an appropriate similarity transformation we may assume with no loss of generality that $\phi(E_{ii}) = E_{ii}$, $i = 1, 2$. Let $A, B \in M_2(\mathbb{D})$ be any matrices and $i, j \in \{1, 2\}$. We will show that if $A_{ij} = B_{ij}$ then $\phi(A)_{ij} = \phi(B)_{ij}$. Indeed, from

$$E_{ii}AE_{jj} = E_{ii}BE_{jj}$$

we conclude that

$$E_{ii}\phi(A)E_{jj} = \phi(E_{ii})\phi(A)\phi(E_{jj}) = \phi(E_{ii}AE_{jj}) = \phi(E_{ii}BE_{jj}) = E_{ii}\phi(B)E_{jj},$$

and thus, $\phi(A)_{ij} = \phi(B)_{ij}$. It follows that

$$\phi\left(\begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}\right) = \begin{bmatrix} \sigma(\lambda_1) & \tau(\lambda_2) \\ \xi(\lambda_3) & \eta(\lambda_4) \end{bmatrix}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{D},$$

where $\sigma, \tau, \xi, \eta \colon \mathbb{D} \to \mathbb{D}$ are maps with $\sigma(1) = \eta(1) = 1$. From $\phi(0) = 0$ we get that $\sigma(0) = \tau(0) = \xi(0) = \eta(0) = 0$. We know that $\phi(E_{12}) = \mu E_{12}$ for some non-zero $\mu \in \mathbb{D}$. Replacing the map ϕ by

$$A \mapsto \begin{bmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{bmatrix} \phi(A) \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix}$$

we may assume with no loss of generality that ϕ has all the properties obtained so far and $\phi(E_{12}) = E_{12}$. Now, because $\phi(E_{21}) = \delta E_{21}$ for some $\delta \in \mathbb{D}$ we get from $E_{12}E_{21} = E_{11}$ that $\phi(E_{21}) = E_{21}$. For every $\lambda \in \mathbb{D}$ we have

$$\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}.$$

Applying ϕ on both sides of the equality we conclude that $\sigma(\lambda) = \tau(\lambda)$, $\lambda \in \mathbb{D}$. In the same way we prove that $\sigma = \tau = \xi = \eta$. From

$$\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda\mu & 0 \\ 0 & 0 \end{bmatrix}, \quad \lambda, \mu \in \mathbb{D},$$

we conclude that σ is multiplicative. And finally, from

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda + \mu \\ 0 & 1 \end{bmatrix}, \quad \lambda, \mu \in \mathbb{D},$$

we get the additivity of σ. This completes the proof in this special case.

It remains to consider the case when $r = n - 1$. If $n = 2$, then $r = 1$ and we are done by the previous case. So, we may assume that $n \geq 3$. We know that every matrix of rank $\leq n - 2$ is mapped into the zero matrix, while every matrix of rank $n - 1$ is mapped into some rank one matrix. Let P be an idempotent of rank one. Then $I - P$ is of rank $n - 1$, and thus $\phi(I - P)$ is a rank one idempotent. We define a map $\psi \colon P_n^1(\mathbb{D}) \to P_n^1(\mathbb{D})$ by $\psi(P) = \phi(I - P)$, $P \in P_n^1(\mathbb{D})$. By Lemma 2.3, we have $PQ = 0$ if and only if $(I - Q)(I - P)$ is of rank $n - 2$, $P, Q \in P_n^1(\mathbb{D})$. The product of two $n \times n$ matrices both of rank $n - 1$ is either of rank $n - 2$, or of rank $n - 1$. Thus, $\phi((I - Q)(I - P)) = 0$ if and only if $PQ = 0$. In other words, for every pair of idempotents $P, Q \in P_n^1$ the following two statements are equivalent:

- $PQ = 0$,
- $\psi(Q)\psi(P) = 0$.

It follows that $P \perp Q$ if and only if $\psi(P) \perp \psi(Q)$. As in the case when $r = 1$ this yields the injectivity of ψ and thus, we can apply [15, Theorem 4.7]. According to this theorem we have two possibilities for ψ and as in the proof of the special case $r = 1$ we can apply the equivalence of the above two conditions to rule out one of the possibilities, thus coming to the conclusion that there exist a non-singular matrix $T \in M_n(\mathbb{D})$ and a non-zero anti-endomorphism $\sigma\colon \mathbb{D} \to \mathbb{D}$ such that

$$\psi(P) = T^t(P^\sigma)T^{-1}, \quad P \in P_n^1(\mathbb{D}).$$

As before, we may assume with no loss of generality that

$$\psi(P) = \phi(I - P) = {}^t(P^\sigma), P \in P_n^1(\mathbb{D}).$$

For $A \in M_n(\mathbb{D})$ and $i, j \in \{1, \ldots, n\}$ we denote by $A(i,j)$ the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and the jth column. In the next step we will show that for every $A \in M_n(\mathbb{D})$ and every pair of integers i, j, $1 \le i, j \le n$, the (i,j)-entry of $\phi(A)$ depends only on $A(i,j)$. This follows from

$$\begin{aligned} \phi(A)_{ij}E_{ij} &= E_{ii}\phi(A)E_{jj} = \phi(I - E_{ii})\phi(A)\phi(I - E_{jj}) \\ &= \phi((I - E_{ii})A(I - E_{jj})). \end{aligned} \tag{6}$$

Thus, there exist maps $\phi_{ij}\colon M_{n-1}(\mathbb{D}) \to \mathbb{D}$, $1 \le i, j \le n$, such that for every $A \in M_n(\mathbb{D})$ we have

$$\phi(A)_{ij} = \phi_{ij}(A(i,j)), \quad 1 \le i, j \le n.$$

Next, we will show that if for some $A \in M_n(\mathbb{D})$ and some pair of integers i, j, $1 \le i, j \le n$, the matrix $A(i,j)$ is singular, then $\phi(A)_{ij} = 0$. Indeed, if $A(i,j)$ is singular, then $(I - E_{ii})A(I - E_{jj})$ is of rank at most $n - 2$, and because ϕ maps matrices of rank at most $n - 2$ into the zero matrix, we get from (6) that $\phi(A)_{ij}$ is zero.

In particular,

$$\phi\left(\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & \ldots & 0 \\ \lambda_2 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \lambda_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{D}$. We know that

$$\phi\left(\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & -1 & -1 & \dots & -1 & 1 \end{bmatrix}.$$

Applying ϕ to both sides of the equality

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 1 \\ 0 & 0 & 1 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

we get that

$$\phi\left(\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}\right) = \pm \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Let $A \in M_{n-1}(\mathbb{D})$ be an arbitrary matrix. We know that the $n \times n$ matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix},$$

where the zeroes denote the zero matrices of the sizes 1×1, $1 \times (n-1)$, and $(n-1) \times 1$, respectively, is mapped by ϕ into $\phi_{11}(A)E_{11}$. Similarly, the matrix

$$\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix},$$

where

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and I is the $(n-2) \times (n-2)$ identity matrix, is mapped into $\phi_{21}(A)E_{21}$. On the other hand, this matrix is mapped into

$$\phi\left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}\right) \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right) = \pm \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \phi_{11}(A)E_{11} = \pm\phi_{11}(A)E_{21}.$$

Thus, $\phi_{21} = \pm\phi_{11}$. In the same way we prove first that $\phi_{21} = \pm\phi_{22}$, and then $\phi_{ij} = \pm\phi_{11}$, $1 \leq i, j \leq n$.

We have

$$\begin{aligned} \phi_{11}(AB)E_{11} = \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & AB \end{bmatrix}\right) &= \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right) \phi\left(\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}\right) \\ &= \phi_{11}(A)\phi_{11}(B)E_{11}, \quad A, B \in M_{n-1}(\mathbb{D}). \end{aligned}$$

Thus ϕ_{11} is multiplicative. By what we have proved at the beginning of this section, it must be degenerate. So, there exists a mulitiplicative map $\xi\colon \overline{\mathbb{D}} \to \mathbb{D}$ such that $\phi_{11}(A) = \xi(\det A(1,1))$, $A \in M_n(\mathbb{D})$. Consequently, $\phi_{ij}(A) = \pm\xi(\det A(i,j))$, $A \in M_n(\mathbb{D})$, $1 \leq i, j \leq n$.

Clearly, if both $\mu_1, \mu_2 \in \mathbb{D}$ belong to the range of ξ, then they commute. It follows that the set of all (i,j)-entries of matrices $\phi(A)$, $A \in M_n(\mathbb{D})$, is a commutative subset of $\mathbb{D}$. On the other hand, $\phi(E_{11} + \cdots + E_{n-1,n-1} + \lambda E_{1n}) = E_{nn} - \sigma(\lambda)E_{n1}$, $\lambda \in \mathbb{D}$. Assume first that $\mathbb{D}$ is not commutative. Then take $\lambda, \mu \in \mathbb{D}$ such that $\lambda\mu - \mu\lambda \neq 0$. As σ is non-zero we have $\sigma(\mu)\sigma(\lambda) - \sigma(\lambda)\sigma(\mu) \neq 0$, and thus the set of $(n,1)$-entries of the matrices $\phi(E_{11} + \cdots + E_{n-1,n-1} + \lambda E_{1n})$, $\lambda \in \mathbb{D}$, is not commutative, a contradiction.

Thus, this last special case that we are treating can occur only if $\mathbb{D} = \mathbb{F}$ is commutative. In this case we know that the (i,j)-entry of $\phi(A)$ is $\pm\xi(\det A(i,j))$, where now det stands for the usual determinant. We further know how ϕ acts on the set of all idempotents P of rank $n-1$. Because the (i,j)-entry of $\phi(P)$ is the corresponding cofactor of the matrix P^σ (note that in the commutative case anti-endomorphisms are actually endomorphisms), we conclude that every A is mapped by ϕ into the matrix of the cofactors of A^σ. This completes the proof of our main theorems. ∎

Remark: Note that we could avoid the use of [15, Theorem 4.7] in the proof of the special case when $r = 1$. However, in the case when $r = n - 1$ our proof essentially depends on this theorem.

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